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IDEALS ASSOCIATED TO BOOLEAN ALGEBRAS OF PROJECTIONS

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Let E be a Banach space and let $\mathscr{B}$ be a Boolean algebra of projections on E'. A $\mathscr{B}$-ideal of E is any closed subspace I of E such that I^0 is the image of a projection in $\mathscr{B}$. This notion brings together various types if ideals such as M-ideals, lattice ideals, L-summands etc. $\mathscr{B}$-ideals are closed under finite sums and finite intersections. They are also mutually orthogonal modulo intersection. If $\mathscr{B}$ is Bade complete, every closed subspace contains a largest $\mathscr{B}$-ideal. Classical results such as de Branges' Lemma, Bishop-Silov Decomposition and Dauns-Hofmann Theorem are extended to this context.

1. THE NOTION OF A $\mathscr{B}$-IDEAL

Let E be a Banach space.

A *Boolean algebra of projections* in E is a family $\mathscr{B}$ of commuting contractive idempotents in $L(E, E)$ which is partially ordered with respect to range inclusion and which is a Boolean algebra with respect to the lattice operations defined by setting

$$P \vee Q = P + Q - PQ$$

$$P \wedge Q = PQ$$

for all $P, Q \in \mathscr{B}$. It is assumed that the unit element of $\mathscr{B}$ is the identity operator $\mathbb{1}$.

$\mathscr{B}$ is said to be *Bade complete* provided that for every family $(P_\alpha)_\alpha$ of elements of $\mathscr{B}$ there exist $\bigvee_\alpha P_\alpha$ and $\bigwedge_\alpha P_\alpha$ in $\mathscr{B}$ and moreover

$$(\bigvee_\alpha P_\alpha)(E) = \overline{\operatorname{Span} \bigcup_\alpha P_\alpha(E)}$$

$$(\bigwedge_\alpha P_\alpha)(E) = \bigcap_\alpha P_\alpha(E).$$

The basic examples are :

$\mathbb{P}_L(E)$, the Boolean algebra of all *L-projections* P on the Banach space E (i.e., $P^2 = P$ and $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in E$) ;

$\mathbb{P}_M(E)$, the Boolean algebra of all *M-projections* P on the Banach space E (i.e., $P^2 = P$ and $\|x\| = \|Px\| \vee \|x - Px\|$ for all $x \in E$) ;

$\mathbb{P}_\wp(E)$, the Boolean algebra of all *band projections* P on the Banach lattice E (i.e., $P^2 = P$ and $0 \leqslant P \leqslant \mathbb{1}$). A natural generalization is obtained by considering instead of Banach lattices regularly ordered Banach spaces (in the sense of Davies [10]).

$\mathbb{P}_{\mathscr{A}}(H)$, the Boolean algebra of all *orthogonal projections* belonging to a commutative von Neumann algebra $\mathscr{A}$ of $L(H, H)$. Here H denotes a Hilbert space.

$\mathbb{P}_L(E)$ and $\mathbb{P}_{\mathscr{A}}(H)$ are also examples of Bade complete Boolean algebras. $\mathbb{P}_{\mathscr{o}}(E)$ is Bade complete provided that E is a Banach lattice with order continuous norm. See [2] and respectively [26] for details.

$\mathbb{P}_L(E)$ and $\mathbb{P}_M(E)$ were first considered by Cunningham [7], [8]. Alfsen-Effros theory (as described in [21], [22]) unifies all above types of projections under the term of a *Cunningham projection*. An important feature of a Cunningham projection P is to be a *mirror projection* i.e.,

$$\|P\| \leqslant 1 \quad \text{and} \quad \|2P - \mathbb{1}\| \leqslant 1.$$

1.1. *Definition*. Let E be a Banach space and let $\mathscr{B}$ be a Boolean algebra of projections on E'. A closed subspace I of E is said to be a *$\mathscr{B}$-ideal* provided that its polar I^0 is the image of a projection P in $\mathscr{B}$.

For $\mathscr{B} = \mathbb{P}_L(E')$ we retrieve the case of M-ideals, first considered by Alfsen and Effros [2]. If E is a C^*-algebra, its M-ideals are precisely the closed two-sided ideals. See [2], p. 167.

For E a regularly ordered Banach space and $\mathscr{B} = \mathbb{P}_{\mathscr{o}}(E')$ we retrieve the case of hypostrict order ideals. See [6], [25]. They coincide with the closed lattice ideals when E is a Banach lattice. Apparently no intrinsic characterization of hypostrict order ideals is known in the general case.

We shall call the $\mathbb{P}_M(E')$-ideals also *L-ideals*. They coincide with the images of L-projections. In fact, if I is an L-ideal of E. Then its polar is the image of an M-projection $P \in L(E', E')$. A result due to Cunningham, Effros and Roy [9] asserts that $\operatorname{Im} P = P(E')$ and $\operatorname{Ker} P = (\mathbb{1} - P)(E')$ are w'-closed as images of M-projections in a dual space. Then $\mathbb{1} - P$ is the adjoint of a projection $Q = L(E, E)$ whose image is I. The fact that Q is indeed an L-projection follows from the duality between L- and M-projections. See [3], p. 15.

If I is a $\mathscr{B}$-ideal, then there exists only one projection P in $\mathscr{B}$ (denoted by P_I) such that $\operatorname{Im} P = I^0$. This is a consequence of the following

1.2 LEMMA. *Let P and Q be two projections in $\mathscr{B}$ such that* $\operatorname{Im} P = \operatorname{Im} Q$. *Then* $P = Q$.

Proof. In fact, for every $x \in E$ there exists a $y \in E$ such that $Px = Qy$. Then $(\mathbb{1} - Q)Px = (\mathbb{1} - Q)Qy = 0$ and in the same manner we can prove that $(\mathbb{1} - P)Q = 0$. Since $PQ = QP$, it follows that $P = QP = PQ = Q$. ■

We shall denote by $\mathscr{I}_{\mathscr{B}}(E)$ the set of all $\mathscr{B}$-ideals of E. Then the map $I \to P_I$ (from $\mathscr{I}_{\mathscr{B}}(E)$ onto $\mathscr{B}$) is bijective and thus $\mathscr{I}_{\mathscr{B}}(E)$ can be organized as a Boolean algebra. We put

$$I^{0\perp} = \operatorname{Im}(\mathbb{1} - P_I) \text{ for } I \in \mathscr{I}_{\mathscr{B}}(E).$$

Let us call *$\mathscr{B}$-summands* of E the images of projections $P \in L(E, E)$ such that $P' \in \mathscr{B}$. Then a remark above asserts that all $\mathbb{P}_M(E')$-ideals are $\mathbb{P}_M(E')$-summands. The same is true for $\mathbb{P}_{\mathscr{A}}(H')$ and $\mathbb{P}_o(E')$ (when the norm of E is order continuous).

A criterion to decide when a $\mathscr{B}$-ideal is a $\mathscr{B}$-summand is indicated in the following proposition:

1.3 Proposition. *Let I be a $\mathscr{B}$-ideal of E. I is a $\mathscr{B}$-summand if and only if there exists a (unique) $\mathscr{B}$-ideal J such that $E = I + J$ and $I \cap J = 0$.*

We shall call J the ($\mathscr{B}$-) *complement* of I.

Proof. The necessity is clear. The sufficiency. Since every $x \in E$ can be uniquely represented as a sum $u + v$ with $u \in I$ and $v \in J$, the mapping $S : x \to u$ is a projection (from E into E) and for every $x' \in E'$, $u \in I$ and $v \in J$ we have

$$x'(S(u + v)) = x'(u) = (P_I x')(u) + (P_J x')(u) =$$

$$= (P_J x')(u) = (P_J x')(u + v)$$

and thus $S \in L(E, E)$ and $S' = P_J$. ■

2. ALGEBRAIC PROPERTIES OF $\mathscr{B}$-IDEALS

In the sequel we shall show that the $\mathscr{B}$-ideals play all nice properties of order ideals and M-ideals.

2.1 Proposition. *Let I and J be two $\mathscr{B}$-ideals of E. Then $I^0 + J^0$ is a w'-closed subspace of E'.*

Proof. First notice that $S = P_I + P_J - P_I P_J$ is a projection in $\mathscr{B}$ whose image is $I^0 + J^0$. In fact, $\operatorname{Im} S \subset I^0 + J^0$ and for $u' \in I^0$ and $v' \in J^0$ we have

$$S(u' + v') = u' + P_J u' - P_J u' + P_I v' + v' - P_I v' =$$

$$= u' + v',$$

by using the commutativity of the projections in $\mathscr{B}$.

Let $x' = Sx'$ with $\|x'\| \leqslant 1$. Then $x' = P_I x' + P_J(\mathbb{1} - P_I)x' \in \in I^0 \cap K + J^0 \cap K$ and thus

$$(I^0 + J^0) \cap K \subset I^0 \cap K + J^0 \cap K \subset I^0 + J^0.$$

Here K denotes the closed unit ball of E'.

The set $(I^0 + J^0) \cap K$ is w'-closed in $I^0 + J^0$ and $I^0 \cap K + J^0 \cap K$ is w'-compact. Then the set $(I^0 + J^0) \cap K$ is w'-compact, so by a well known result due to Krein and Smulian (see [12], vol. I, p. 434), the set $I^0 + J^0$ is w'-closed. ■

2.2 COROLLARY. *The intersection of a finite family of $\mathscr{B}$-ideals is still a $\mathscr{B}$-ideal.*

Proof. Let I and J be two $\mathscr{B}$-ideals. Then $(I \cap J)^0$ is the w'-closure of $I^0 + J^0$, so by Proposition 2.1 it coincides with $I^0 + J^0$. ∎

In contrast to the situation for ideals in rings, arbitrary intersections of M-ideals need not be M-ideals, so we cannot expect a better result in the general case. See [5].

We shall say that a Banach space E is *$\mathscr{B}$-distinguished* provided that every intersection of $\mathscr{B}$-ideals of E is also a $\mathscr{B}$-ideal.

Every C^*-algebra $\mathscr{A}$ is $\mathbb{P}_L(\mathscr{A}')$-distinguished and every Banach lattice E is $\mathbb{P}_o(E')$-distinguished. Every G-space E is $\mathbb{P}_L(E')$-distinguished. See [27].

2.3 PROPOSITION. *Let I and J be two $\mathscr{B}$-ideals of E. Then $I + J$ is a closed subspace of E.*

Proof. Assume first that $I \cap J = 0$. Since every $x \in X = I + J$ can be uniquely represented as $x = u + v$ with $u \in I$ and $v \in J$, we can consider the linear projection $S : x \to u$ from X into X.

Notice that $X' = (I + J)' = E'/I^0 \cap J^0 = (I^0 \cap J^0)^\perp = \mathrm{Im}(\mathbb{1} - P_I P_J)$ and X' is the direct sum of $X' \cap I^0$ and $X' \cap J^0$. Put $P_1 = (P_I - P_I P_J)|X'$ and $P_2 = (P_J - P_I P_J)|X'$. Then

$$x'(S(u + v)) = x'(u) = P_1 x'(u) = P_2 x'(u) =$$

$$= P_2 x'(u) = P_2 x'(u + v)$$

for all $x' \in X'$, $u \in I$, $v \in J$ and thus $S \in L(X, X)$ and $\|S\| \leq 1$.

Let $(z_n)_n$ be a Cauchy sequence in X. For each n there exists a unique decomposition $z_n = v_n + v_n$ with $u_n \in I$ and $v_n \in J$. Then $u_n = Sz_n$ $(n \in \mathbb{N})$ is a Cauchy sequence in I and thus converging to a $u \in I$. Clearly, $v = \lim_{n\to\infty} v_n$ exists and belongs to J, so that $\lim_{n\to\infty} z_n = u + v$.

If I and J are not orthogonal, we consider the closed ideal $H = I \cap J$. From what we have shown it follows that $\pi(I) + \pi(J)$ (π denotes the canonical mapping $E \to E/H$) is closed in E/H and thus $I + J = \pi^{-1}(\pi(I) + \pi(J))$ is closed in E. ∎

2.4 COROLLARY. *If $I_1, \ldots, I_n$ are $\mathscr{B}$-ideals of E then $I_1 + \ldots + I_n$ is also a $\mathscr{B}$-ideal.*

Proof. It suffices to consider the case where $n = 2$. Then $I_1 + I_2$ is a closed subspace and $(I_1 + I_2)^0 = I_1^0 \cap I_2^0$ is the image of $P_{I_1 I_2} \in \mathscr{B}$. ∎

2.5 PROPOSITION. *Suppose that $\mathscr{B}$ is a Bade complete Boolean algebra of projections and $(I_\alpha)_\alpha$ is an arbitrary family of $\mathscr{B}$-ideals. Then* $\overline{\mathrm{Span}\bigcup_\alpha I_\alpha}$ *is a $\mathscr{B}$-ideal.*

Proof. In fact, $(\overline{\mathrm{Span}\bigcup_\alpha I_\alpha})^0 = \bigcap_\alpha I^0$. ∎

2.6 COROLLARY. *Under the hypothesis of Proposition 2.5, in every closed subspace of E there exists a largest $\mathscr{B}$-ideal contained in it.*

If I is a $\mathscr{B}$-ideal of E then $\mathscr{B}|I^0 = \{PP_I \mid P \in \mathscr{B}\}$ is a Boolean algebra of projections on $(E/I') = I^0$ and $\mathscr{B}|I^{0\perp} = \{P(\mathbb{1} - P_I) \mid P \in \mathscr{B}\}$ is a Boolean algebra of projections on $I' = I^{0\perp}$; both are Bade complete provided that $\mathscr{B}$ is. Then we can speak on the $\mathscr{B}|I^{0\perp}$-ideals of I and $\mathscr{B}|I^0$-ideals of E/I, called also the $\mathscr{B}$-ideals of I and respectively E/I.

2.7 PROPOSITION. *Suppose that I is a $\mathscr{B}$-ideal of E and let $\pi : E \to E/I$ be the canonical mapping.*

i) *The $\mathscr{B}$-ideals of I are precisely the $\mathscr{B}$-ideals of E contained in I.*

ii) *The images by π of $\mathscr{B}$-ideals are $\mathscr{B}$-ideals and the inverse images of $\mathscr{B}$-ideals are $\mathscr{B}$-ideals that contain I.*

The next result shows that the $\mathscr{B}$-ideals are mutually orthogonal modulo intersection.

2.8 PROPOSITION. *Let I and J be two $\mathscr{B}$-ideals of E. Then the canonical images of I and J in $(I + J)/I \cap J$ are complementary summands.*

Proof. By Proposition 2.3, we can assume that $I + J = E$. Then $E/I \cap J$ is the algebraic sum of canonical images of I and J. Since

$$(I/I \cap J)^0 = (I^0 \cup J^0) \cap I^0 = \operatorname{Im}(P_I + P_J)\, P_I$$

it follows that $I/I \cap J$ is a $\mathscr{B}$-ideal. A similar argument works for $J/I \cap J$. The proof ends with an appeal to Proposition 1.3. ∎

3. OPERATOR ALGEBRAS ASSOCIATED TO BOOLEAN ALGEBRAS OF PROJECTIONS

Let E be a Banach space over the field $\mathbb{K}$; $\mathbb{K}$ may be $\mathbb{R}$ or $\mathbb{C}$.

To any Boolean algebra $\mathscr{B}$ of projections on E one can associate a commutative Banach algebra with unit, *the Bade algebra* generated by $\mathscr{B}$,

$$\mathscr{C}_{\mathscr{B}}(E) = \overline{\operatorname{Span} \mathscr{B}}$$

the closure being taken in the norm topology of $L(E, E)$. The basic result about $\mathscr{C}_{\mathscr{B}}(E)$ is due to Bade (see[12], vol. III, ch. XVIII):

3.1 THEOREM. *Suppose that $\mathscr{B}$ is Bade complete. Then $\mathscr{C}_{\mathscr{B}}(E)$ consists of all $T \in L(E, E)$ that leave invariant all subspaces that are $\mathscr{B}$-invariant.*

If $\mathscr{B}$ consists only of mirror projections, then $\mathscr{C}_{\mathscr{B}}(E)$ is algebraic and isometric isomorphic to $C(\operatorname{Spec} \mathscr{B}, \mathbb{K})$, where $\operatorname{Spec} \mathscr{B}$ denotes the Stone space associated to $\mathscr{B}$. See Evans [15]. Consequently, in this case $\mathscr{C}_{\mathscr{B}}(E)$ is a Banach lattice (possibly complex). If we denote by $\operatorname{Re}\mathscr{C}_{\mathscr{B}}(E)$ the closure of finite real combinations of elements of $\mathscr{B}$, the above isomorphism induces the isomorphism

$$\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E) \xrightarrow{\sim} C(\operatorname{Spec} \mathscr{B}, \mathbb{R}).$$

Notice that given any Boolean algebra of equicontinuous projections on E, the mapping

$$x \to |||x||| = \sup \{\|Px\|, \|2Px - x\| \mid P \in \mathscr{B}\}$$

is an equivalent norm on E and every projection in $\mathscr{B}$ is a mirror projection on $(E, |||\ \ |||)$.

If $\mathscr{B}$ is a Bade complete Boolean algebra of mirror projections on E, then $\mathscr{C}_{\mathscr{B}}(E)$ is a commutative von Neumann algebra (i.e., a dual $C(S)$-space) and every w'-converging net in $\mathscr{C}_{\mathscr{B}}(E)$ is also wo-converging in $L(E, E)$. See Orhon [24]. Recall that $T_\alpha \xrightarrow{wo} T$ if and only if $x'(T_\alpha x) \to x'(Tx)$ for every $x' \in E'$ and every $x \in E$.

3.2 *Definition*. Let $\mathscr{B}$ be a Boolean algebra of projections on E'. We shall call the algebra

$$Z_{\mathscr{B}}(E) = \{T \mid T \in L(E, E),\ T' \in \mathscr{C}_{\mathscr{B}}(E')\}$$

the *centralizer* associated to $\mathscr{B}$.

$Z_{\mathscr{B}}(E)$ is a commutative Banach algebra with unit $\mathbb{1}$ when endowed with the induced norm.

The notion of a centralizer was previously considered by Alfsen and Effros [2] in the context of M-structure theory and by Wils [29] in the context of regularly ordered Banach spaces. See also [19] and [21] for further extensions. The case considered by Alfsen and Effros corresponds to $\mathscr{B} = \mathbb{P}_L(E')$ and their centralizer is denoted by $Z_M(E)$. See Theorem 4.2 in [2] for the agreement with the original definition of $Z_M(E)$. Notice that Alfsen and Effros considered only real Banach spaces.

If $\mathscr{A}$ denotes a C^*-algebra with unit, then $Z_M(\mathscr{A})$ is just the space of all operators $M_z : x \to zx$, where z runs over all elements of the centre of $\mathscr{A}$. See [2]. This example motivates the terminology in Definition 3.2 above.

For E a Banach lattice and $\mathscr{B} = \mathbb{P}_o(E')$, the corresponding centralizer is denoted by $Z_o(E)$. We shall show that $Z_o(E)$ coincides with Wils' centralizer of E,

$$WZ_o(E) = \{T \mid T \in L(E, E),\ -\alpha \cdot \mathbb{1} \leqslant T \leqslant \alpha \cdot \mathbb{1} \text{ for some } \alpha \geqslant 0\}.$$

For, we need a preparation.

3.3 Lemma (Wils [29]). *$WZ_o(E)$ is a Banach lattice with a strong order unit $\mathbb{1}$ and the induced norm on $WZ_o(E)$ coincides with the norm*

$$\|\ \|_{\mathbb{1}} : T \to \inf\{\alpha \mid -\alpha \cdot \mathbb{1} \leqslant T \leqslant \alpha \cdot \mathbb{1}\}.$$

Particularly, $WZ_o(E)$ can be viewed as a space $C(S, \mathbb{R})$.

3.4 Lemma. *Suppose that E is an order complete Banach lattice. Then $WZ_o(E)$ is also an order complete Banach lattice that coincides with*

$$\mathscr{C}_{\mathscr{B}}(E) \text{ for } \mathscr{B} = \mathbb{P}_o(E).$$

Proof. Because E is order complete, the regular operators $T \in L(E, E)$ constitute an order complete Banach lattice $L_r(E, E)$ such that every increasing and majorized net of positive operators is pointwise converging to its l.u.b. Or, $WZ_o(E) \subset L_r(E, E)$ and $\|\ \| = \|\ \|_{\mathbb{1}}$ on $WZ_o(E)$, so that $WZ_o(E)$ is order complete.

Clearly, $\mathscr{C}_{\mathscr{B}}(E) \subset WZ_o(E)$ for $\mathscr{B} = \mathbb{P}_o(E)$. The two spaces have the same idempotents, the elements of $\mathbb{P}_o(E)$. To prove the other inclusion is more convenient to identify $WZ_e(E)$ with a space $C(S, \mathbb{R})$. Since $WZ_e(E)$ is order complete, S is totally disconnected i.e., the closure of every open subset of S is also open. Consequently the algebra Span $\mathbb{P}_e(E)$, generated by the characteristic functions of open-and-closed subsets of S, separates S. By Stone-Weierstrass Theorem, this algebra is norm dense in $C(S, \mathbb{R})$ and thus $\mathscr{C}_{\mathscr{B}}(E) = WZ_e(E)$ for $\mathscr{B} = \mathbb{P}_e(E)$.

3.5 Lemma (Wickstead [28]). *An operator $T \in L(E, E)$ belongs to $WZ_e(E)$ if and only if it leaves invariant every lattice ideal of E.*

By Lemma 3.5,

$$Z_e(E) \subset WZ_e(E).$$

If $T \in WZ_e(E)$, then $T' \in WZ_e(E') = \mathscr{C}_{\mathscr{B}}(E')$ for $\mathscr{B} = \mathbb{P}_e(E')$ i.e., $T \in Z_e\,(E')$. Consequently.

$$Z_e(E) = WZ_e(E).$$

Notice that in general every $T \in Z_{\mathscr{B}}(E)$ leaves invariant every $\mathscr{B}$-ideal. If we define the real part of the centralizer by

$$\operatorname{Re} Z_{\mathscr{B}}(E) = \{T | T \in L(E, E),\ T' \in \operatorname{Re} \mathscr{C}_{\mathscr{B}}(E')\}$$

and $\mathscr{B}$ consists only of mirror projections then $\operatorname{Re} Z_{\mathscr{B}}(E)$ constitutes a Banach lattice with respect to the order relation

$$S \leqslant T \text{ in } \operatorname{Re} Z_{\mathscr{B}}(E) \text{ if and only if } S' \leqslant T' \text{ in}$$

$$\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E') = C(\operatorname{Spec} \mathscr{B}, \mathbb{R}).$$

Alfsen-Effros theory describes the order relations on $\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E)$ via suitable order relations on E. We shall illustrate that in a very special case, first considered in [2], namely for

$$\mathscr{C}_L(E) = \mathscr{C}_{\mathscr{B}}(E), \text{ where } \mathscr{B} = \mathbb{P}_L(E).$$

The L-order relation on a Banach space E is given by

$$x \ll_L y \text{ if and only if } \|y\| = \|x\| + \|y - x\|.$$

Then $0 \leqslant S \leqslant T$ in $\operatorname{Re} \mathscr{C}_L(E)$ if and only if $Sx \ll_L Tx$ for every $x \in E$.

3.6 PROPOSITION. *Suppose that $\mathscr{B}$ is Bade complete and let $T \in \operatorname{Re} Z_{\mathscr{B}}(E)$ such that $0 \leqslant T \leqslant \mathbb{1}$. Then* Im *$T$ is a $\mathscr{B}$-ideal.*

Proof. By hypothesis, $0 \leqslant T' \leqslant \mathbb{1}$ in $\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E')$. We have to show that $(\operatorname{Im} T)^0 = \operatorname{Ker} T'$ is the image of a projection of $\mathscr{B}$.

Via a renormalization process, we can assume that $\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E') = C(\operatorname{Spec} \mathscr{B}, \mathbb{R})$. Since this space is order complete, there exists an increasing sequence of finite linear combinations of elements of $\mathscr{B}$ with positive coefficients such that $\|S_n - T'\| \to 0$. Then $T' = \sup_n S_n$ in $\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E')$ and $\bigcap \operatorname{Ker} S_n \subset \operatorname{Ker} T'$. Actually equality holds because the functionals $A \to (Ax')x$ $(x \in E,\ x' \in E')$ are order continuous and separates $\operatorname{Re} \mathscr{C}_{\mathscr{B}}(E')$. See [24]. Each $\operatorname{Ker} S_n$ is a finite intersection of images of projections in $\mathscr{B}$. Since $\mathscr{B}$ is Bade complete, the above reasoning shows that $\operatorname{Ker} T'$ is also the image of a projection in $\mathscr{B}$. ■

4. THE BISHOP-SILOV DECOMPOSITION

In this section we discuss a class of ideals important in approximation theory.

We first briefly survey some basic facts on function algebras. The details will be found in Gamelin [16].

Let K be a compact Hausdorff space. By a function algebra on K we mean any closed subalgebra $\mathscr{A}$ of $C(K, \mathbb{C})$ that contains the constants and separates K. A subset H of K is a *set of anti-symmetry* of $\mathscr{A}$ provided that

$$f \in \mathscr{A} \text{ and } f \mid H \text{ is a real function implies } f \mid H \text{ is constant.}$$

The algebra $\mathscr{A}$ is said to be *anti-symmetrical* provided that K itself is a set of anti-symmetry.

The closure of a set of anti-symmetry so is a set of anti-symmetry. Every point $x \in K$ belongs to a maximal set of anti-symmetry and every maximal set anti-symmetry is closed. The next two propositions below concentrate basic facts on the concept of anti-symmetry :

4.1 de BRANGES' LEMMA. *Let $\mathscr{A}$ be a function algebra over K and let μ be an extreme point of the unit ball of $\mathscr{A}^0$. Then* Supp μ, *the support of* μ, *is a set of anti-symmetry of $\mathscr{A}$.*

4.2 THEOREM. (The Bishop-Silov decomposition). *Let $\mathscr{A}$ be a function algebra on K. Then K admits a decomposition $K = \bigcup K_\alpha$ where $(K_\alpha)_0$ is the set of all maximal subsets of anti-symmetry. Then $K_\alpha \cap K_\beta = \emptyset$ for $\alpha \neq \beta$ and moreover*

a) $\mathscr{A} \mid K_\alpha = \{f \mid K_\alpha \mid f \in \mathscr{A}\}$ *is a closed subspace of* $C(K_\alpha, \mathbb{C})$ *for every* α;

b) $f \in C(K, \mathbb{C})$ *belongs to $\mathscr{A}$ if and only if $f \mid K_\alpha \in \mathscr{A} \mid K_\alpha$ for every* α.

We shall extend all these facts by noticing the existence of an one-to-one correspondence between the closed subsets H of K and the closed M-ideals of $C(K, \mathbb{C})$,

$$H \to I_H = \{f \mid f \in C(K, \mathbb{C}),\ f \mid H = 0\}.$$

Moreover, $C(K, \mathbb{C})/I_H$ is isometric to $C(H, \mathbb{C})$ and thus the restriction mapping $f \to f \mid H$ coincides with the canonical mapping $C(K, \mathbb{C}) \to C(K, \mathbb{C})/I_H$. The elements of $C(K, \mathbb{C})$ can be seen naturally as members of $Z_M(C(K, \mathbb{C}))$ via the isomorphism

$$M : C(K, \mathbb{C}) \to Z_M(C(K, \mathbb{C})),\quad M(f)\,g = fg.$$

In the sequel E will denote a complex Banach space, $\mathcal{B}$ a Boolean algebra of projections on E' and X a subspace of E.

4.3 *Definition*. A $\mathcal{B}$-ideal I of E is said to be *anti-symmetric* with respect to X provided that every $U \in \mathrm{Re}\, Z_{\mathcal{B}|I^0}\ (E/I)$ such that $U(\pi_I\ (X)) \subset \subset \pi_I(X)$ is a multiple of $\mathbb{1}_{E/I}$.

The notion of a set of anti-symmetry represents the case where $E = C(K, \mathbb{C})$ and $\mathcal{B} = \mathbb{P}_L(E')$. For $X = 0$, Definition 4.3 reduces to the notion of a $\mathcal{B}$-ideal.

We shall denote by $\mathcal{A}_{\mathcal{B},X}(E)$ the set of all anti-symmetric $\mathcal{B}$-ideals of E, with respect to X. Clearly,

$$\mathcal{A}_{\mathcal{B},X}(E) \subset \mathcal{A}_{\mathcal{B},\overline{X}}(E).$$

4.4 Lemma. *Suppose that $\mathcal{B}$ is Bade complete and E is $\mathcal{B}$-distinguished. Let $(I_\alpha)_\alpha \subset \mathcal{A}_{\mathcal{B},X}(E)$ such that $J = \overline{\mathrm{Span}}\ (\cup\, I_\alpha) \neq E$. Then*

$$I = \cap\, I_\alpha \in \mathcal{A}_{\mathcal{B},\overline{X}}(E).$$

Proof. We start by noticing the existence of a canonical mapping relating the centralizers. In order to simplify the notation we shall omit the appropriate indices of Z.

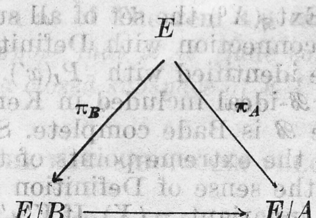
Let A and B two $\mathcal{B}$-ideals of E such that $B \subset A$. Then we can consider the mapping

$$M_{BA} : Z(E/B) \to Z(E/A)$$

given by

$$M_{BA}(U)\,(\pi_A(x)) = \pi_{BA}(U\pi_B(x))$$

where π_{BA} makes commutative the diagramm



and π_A and π_B are the canonical quotient mappings. Clearly, M_{BA} maps $\mathrm{Re}\, Z(E/B)$ into $\mathrm{Re}\, Z(E/A)$.

To prove our Lemma, let $U \in \mathrm{Re}\, Z(E/I)$ such that $U(\pi_I(X)) \subset \pi_I(X)$. Then for each α,

$$M_{I_\alpha I}(U)(\pi_{I_\alpha}(X)) = \pi_{I_\alpha I}(U\ \pi_I(X)) \subset \pi_{I_\alpha I}(\pi_I(X)) =$$

$$= \pi_{I_\alpha}(X).$$

Since $I_\alpha \in \mathscr{A}_{\mathscr{B},X}(E)$, there exists an $a_\alpha \in \mathbb{R}$ such that $M_{I_\alpha I}(U) = a_\alpha \cdot \mathbb{1}_{E/I_\alpha}$ and thus

$$M_{IJ}(U) = (M_{I_\alpha J} \circ M_{II_\alpha})(U) = a_\alpha \cdot \mathbb{1}_{E/J}.$$

As $E/J \neq 0$, it follows that $a_\alpha = a_\beta = a$ for all α and β. Then $M_{II_\alpha}(U - a \cdot \mathbb{1}_{E/I}) = 0$ for all α and thus $U = a \cdot \mathbb{1}_{E/I}$ i.e., $I \in \mathscr{A}_{\mathscr{B},X}(E)$. ∎

4.5 COROLLARY. *Under the hypothesis of Lemma 4.4, every $I \in \mathscr{A}_{\mathscr{B},X}(E)$, $I \neq E$, contains a (unique) minimal ideal I_0 in $\mathscr{A}_{\mathscr{B},X}(E)$.*

We shall denote by $\tilde{\mathscr{A}}_{\mathscr{B},X}(E)$ the set of all minimal ideals in $\mathscr{A}_{\mathscr{B},X}(E)$. $\tilde{\mathscr{A}}_{\mathscr{B},X}(E)$ can be viewed as an analogue of Bishop-Silov decomposition. In fact, the following result was proved in [23]:

4.6 THEOREM (Bishop-Silov decomposition for $\mathbb{P}_L(E')$-distinguished spaces). *Suppose that E is $\mathbb{P}_L(E')$-distinguished and X is a closed subspaces of E. Then:*

a) *$\pi_I(X)$ is a closed subspace of E/I for every $I \in \tilde{\mathscr{A}}_{\mathbb{P}_L(E'),X}(E)$.*

b) *For every $x \in E$,*

$$d(x, X) = \sup \{d(\pi_I(x), \pi_I(X)) \mid I \in \tilde{\mathscr{A}}_{\mathbb{P}_L(E'),X}(E)\}.$$

Bishop-Silov decomposition for $\mathscr{B} = \mathbb{P}_L(E')$ depends essentially on two facts: i) the possibility to extend de Branges'Lemma, and ii) the fact that the unit ball of E' contains sufficiently many extreme points (as stated by Krein-Milman Theorem). The problem of defining a good notion of extremality directly in terms if $\mathscr{B}$ seems rather delicate and our option is merely tentative.

4.7 *Definition.* Let E, $\mathscr{B}$ and X as above. A norm 1 element x' of X^0 is said to be a *$\mathscr{B}$-extreme point of the unit ball of X^0* provided that if I is a $\mathscr{B}$-ideal included in Ker x', then y', the lifting of x' to E/I, is an eigenvector for the adjoint of every $V \in Z_{\mathscr{B}/I^0}(E/I)$ which leaves invariant $\pi_I(X)$.

We shall denote by $\mathrm{Ext}_{\mathscr{B}}(X^0)$ the set of all such points.

Some comments in connection with Definition 4.7 are in order. Notice first that y' can be identified with $P_I(x')$. Also, we may restrict ourselves to the maximal $\mathscr{B}$-ideal included in Ker x'(if exists); such an ideal exists if for example $\mathscr{B}$ is Bade complete. See Corollary 2.6.

If $\mathscr{B} = \mathbb{P}_L(E')$, then the extreme points of the unit ball of X^0 are also $\mathscr{B}$-extreme points in the sense of Definition 4.7 above. In fact, let $V \in Z_{\mathscr{B}/I^0}(E/I)$ that leaves invariant $\pi_I(X)$. If $V'y' = 0$ or $V'y' = y'$ then the proof is done. In the other case, notice that

$$y' = \|V'y'\| \cdot \frac{V'y'}{\|V'y'\|} + \|y' - V'y'\| \cdot \frac{y' - V'y'}{\|y' - V'y'\|}$$

which yields the following convex combination in X^0,

$$x' = y' \circ \pi_I = \|V'y'\| \cdot \frac{(\pi_I' \circ V')\, y'}{\|V'y'\|} + \|y' - V'y'\| \cdot \frac{\pi_I'(\mathbb{1} - V')\, y'}{\|y' - V'y'\|}.$$

By hypothesis, $x' = (\pi_I' \circ V')(y')/\|V'y'\|$ i.e.,

$$y' = V'(y')/\|V'y'\|$$

so that y' is an eigenvector of V'. ■

The following example shows that $\mathscr{B}$-extreme points can be different of $\mathbb{P}_L(E')$-extreme points. For, consider the case where $E = \ell^1$, $X = 0$, $\mathscr{B} = \mathbb{P}_\circ(E')$. Then each $e_n = (\delta_{jn})_j$ belongs to $\mathrm{Ext}_{\mathscr{B}}(\ell^\infty)$ though it is not an extreme point in the usual sense.

4.8 de BRANGES' LEMMA. *Suppose that $\mathscr{B}$ is Bade complete and x' is a $\mathscr{B}$-extreme point of the unit ball of X^0. Then the maximal $\mathscr{B}$-ideal I contained in* $\mathrm{Ker}\, x'$ *belongs to $\mathscr{A}_{\mathscr{B},X}(E)$.*

Proof. Let $U \in \mathrm{Re}\, Z_{\mathscr{B}|I^0}(E/I)$ such that $0 \leqslant U \leqslant \mathbb{1}$ and $U(\pi_I(X)) \subset \subset \pi_I(X)$.

Since $I \subset \mathrm{Ker}\, x'$, there exists a unique $y' \in (E/I)'$ such that $y' \circ \pi_I = x'$. Since x' is a $\mathscr{B}$-extreme point of the unit ball of X^0, there exists an $\alpha \in \mathbb{R}$ such that

$$(U' - \alpha \cdot \mathbb{1})\, y' = 0.$$

To end the proof we shall show that in general

$$(*) \qquad V \in Z_{\mathscr{B}|I}(E/I),\ V'y' = 0 \text{ implies } V = 0.$$

In fact, by replacing V by V^2 if necessary, we may assume in addition that $0 \leqslant V \leqslant \mathbb{1}$. Due to the selection of I, there exists no $\mathscr{B}$-ideal $J \neq 0$ included in $\mathrm{Ker}\, y'$. Or, $V'y' = 0$ yields $\overline{\mathrm{Im}\, V} \subset \mathrm{Ker}\, y'$ and $\overline{\mathrm{Im}\, V}$ is a $\mathscr{B}$-ideal. See Proposition 3.6 above. Consequently $V = 0$ and thus the assertion $(*)$ is proved.

By $(*)$, $U = \lambda \cdot \mathbb{1}$ for a real λ, which shows that I is anti-symmetric. ■

We end this section by noticing that the argument given in [23] shows that the assertion a) of Theorem 4.6 can be extended as follows

4.9 LEMMA. *Suppose that $\mathscr{B}$ is Bade complete and E is $\mathscr{B}$-distinguished. Then $\pi_I(X)$ is a closed subspace of E/I for each $I \in \tilde{\mathscr{A}}_{\mathscr{B},X}(E)$.*

5. THE DAUNS-HOFMANN THEOREM

Roughly speaking, Dauns-Hofmann Theorem asserts that every bounded continuous scalar-valued function on the spectrum of a C^*-algebra multiplies the algebra. This theorem was generalized by Alfsen and Effros in their work on the structure of real Banach spaces. See Theorem 4.9 in [2]. A further investigation of this result was done by G. A. Elliott [14] (see also [13]) by considering the so called G-primitive structures. It is the purpose of this section to show how his idea work in the framework of $\mathscr{B}$-ideals.

In the sequel E will denote a Banach space and $\mathscr{B}$ a Boolean algebra of projections on E'. It was already noticed that 0 and E belong to $\mathscr{I}_{\mathscr{B}}(E)$

and $\mathscr{I}_{\mathscr{B}}(E)$ is closed under finite sums and finite intersections. Also, the $\mathscr{B}$-ideals are mutually orthogonal modulo intersection. The latter fact can be restated as follows :

5.1 LEMMA. *Let I and J be two $\mathscr{B}$-ideals of E. Then the canonical isomorphism*

$$\Phi : (I + J)/J \to I/I \cap J$$

is an isometry.

Proof. Notice first that $(E/H)' = H^0$ and $H' = E'/H^0$ for every $\mathscr{B}$-ideal H. Then

$$((I + J)/J') = \mathrm{Im}(\mathbb{1} - P_I P_J) \cap \mathrm{Im}\, P_J = I^{0\perp} \cap J^0$$

$$(I/I \cap J)' = I^{0\perp} \cap (I^0 + J^0) = I^{0\perp} \cap J^0$$

so that under these identifications Φ' coincides with the identity of $I^{0\perp} \cap J^0$. Consequently Φ' is an isometric isomorphism and thus Φ itself is an isometric isomorphism. ∎

As in [14], we can infer from Lemma 5.1 the following

5.2 COROLLARY. *Let $I_0, \ldots, I_n$ be $\mathscr{B}$-ideals of E and $x = I_0 + \ldots + I_n$. Then there exist $x_0 \in I_0, \ldots, x_n \in I_n$ such that $x = x_0 + \ldots + x_n$ and $\|x_k\| \leqslant 3\,\|x\|$ for every $k \in \{0, \ldots, n\}$.*

Let there be given a family $\mathrm{Prim}_{\mathscr{B}}(E)$ of proper $\mathscr{B}$-ideals (called *primitive $\mathscr{B}$-ideals*) such that every proper $\mathscr{B}$-ideal is an intersection of primitive $\mathscr{B}$-ideals. $\mathrm{Prim}_{\mathscr{B}}(E)$ is endowed with *the structure pseudo-topology* τ_{str} ; consisting of all complements of hulls (a *hull* being the set of all primitive $\mathscr{B}$-ideals containing some fixed $\mathscr{B}$-ideal). $\mathrm{Prim}_{\mathscr{B}}(E)$ is a T_0-space ; for $I \notin J$ we have $J \notin h(I)$ while $I \in h(I)$.

5.3 *Remark.* If primitive $\mathscr{B}$-ideals are prime i.e.,

$$I_1, I_2 \in \mathscr{I}_{\mathscr{B}}(E) \text{ and } I_1 \cap I_2 \subset I \Rightarrow I_1 \subset I \text{ or } I_2 \subset I,$$

and $\mathscr{B}$ is Bade complete, then τ_{str} is indeed a topology. This is the case if $\mathscr{B} = \mathbb{P}_L(E')$ and $\mathrm{Prim}_{\mathscr{B}}(E)$ consists of all primitive M-ideals of E in the sense of Alfsen and Effros [2].

In fact, if $h(I)$ denotes the hull of I then

$$\bigcap_{\alpha} h(I_\alpha) = h\,(\overline{\mathrm{Span} \bigcup_{\alpha} I_\alpha})$$

by Corollary 2.6 above. Also it is clear that

$$h(I_1) \cup h(I_2) \subset h(I_1 \cap I_2)$$

for every two $\mathscr{B}$-ideals I_1 and I_2. Because primitive $\mathscr{B}$-ideals are suppose to be prime, the other inclusion is also true, so that

$$h(I_1) \cup h(I_2) = h(I_1 \cap I_2).$$

Finally, $\mathrm{Prim}_{\mathscr{B}}(E) = h(0)$ and $\mathcal{O} = h(E)$. ■

A function $f : \mathrm{Prim}_{\mathscr{B}}(E) \to \mathbb{R}$ is said to be *continuous* provided that the inverse image of every open interval is the complement of a hull. If f takes complex values then its continuity will mean that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are both continuous in that sense.

We shall denote by $C_b(\mathrm{Prim}_{\mathscr{B}}(E), \mathbb{K})$ the Banach space of all bounded continuous functions $f : \mathrm{Prim}_{\mathscr{B}}(E) \mapsto \mathbb{K}$ endowed with the sup norm.

5.3 Dauns-Hofmann Generalized Theorem. *Let E be a Banach space over the field $\mathbb{K}$, let $f \in C_b\,(\mathrm{Prim}_{\mathscr{B}}(E), \mathbb{K})$ and $x \in E$. Then there exists a unique element $fx \in E$ such that*

$$(fx)\,(t) = f(t) \cdot x(t) \quad \textit{for every} \quad t \in \mathrm{Prim}_{\mathscr{B}}(E).$$

Here $z(t)$ denotes the class of z in E/t.

Though formally Theorem 5.3 above is a special case of Theorem 3.1 in [14], it seems better suited for applications.

By the closed graph theorem, for each $f \in C_b(\mathrm{Prim}_{\mathscr{B}}(E), \mathbb{K})$ the mapping

$$M_f : x \to fx$$

given by Theorem 5.3, belongs to $L(E, E)$. A second application of the closed graph theorem shows that the mapping $M : f \to M_f$ is also continuous and thus a morphism of Banach algebras from $C_b(\mathrm{Prim}_{\mathscr{B}}(E), \mathbb{K})$ into $L(E, E)$.

We shall call the mappings M_f *multipliers*. By Theorem 5.3, every multiplier leaves invariant the $\mathscr{B}$-ideals and induces a homothety in each quotient E/I with $I \in \mathrm{Prim}_{\mathscr{B}}(E)$.

For real Banach spaces E and $\mathscr{B} = \mathbb{P}_L(E')$, Alfsen and Effros [2] have proved that $Z_M(E)$ equals the algebra of multipliers on E. The case of disc algebra shows that in general not every multiplier is an element of the centralizer.

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BIBLIOGRAPHY

1. Alfsen E., *M-structure and intersection properties of balls in Banach spaces*, Israel, J. Math., **13** (1972), 235–245.
2. Alfsen E. and Effros E. G., *Structure in real Banach spaces I, II*, Annals of Math., **96** (1972), 98–173.
3. Behrends E., *M-structure and the Banach-Stone theorem*, Lecture Notes in Math., vol. **736**, Springer-Verlag, 1979.
4. Bernard A., *Caractérisations de certaines parties d'un espace compact muni d'un espace vectoriel ou d'une algèbre de fonctions continues*, Ann. Inst. Fourier Grenoble, **17**, **2** (1967), 359–382.
5. Bunce J., *The intersection of closed ideals in a simplex space need not be an ideal*, J. London Math. Soc. (**2**), **1** (1969), 67–68.
6. Combes J. F. et Perdrizet F., *Certaines idéaux dans les espaces vectoriels ordonnés*, J. Math. Pures et Appl., **49** (1970), 29–59.

7. Cunningham F. Jr., *L-structure in L-spaces*, Trans. Amer. Math. Soc., **95** (1960), 274–299
8. Cunningham F. Jr., *M-structure in Banach spaces*, Proc. Cambridge Philos. Soc., **63** (1967) 613–639.
9. Cunningham F., Effros E. and Roy N., *M-structure in dual Banach spaces*, Israel J. Math., **14** (1973), 304–308.
10. Davies E. B., *The structure and ideal theory of the predual of a Banach lattice*, Trans. Amer. Math. Soc., **131** (1968), 544–555.
11. Dodds P. G. and Ricker W., *Spectral measures and the Bade reflexivity theorem*, J. Funct. Anal., **61** (1985), 136–163.
12. Dunford N. and Schwartz J., *Linear operators*, Wiley-Interscience, vol. I(1958), vol. II(1963), vol. III(1971).
vol. II (1963), vol. III (1971).
13. Elliott G. A. and Olesen D., *A simple proof of the Dauns-Hofmann theorem*, Math. Scand., **34** (1974), 231–234.
14. Elliott G. A., *An abstract Dauns-Hofmann-Kaplansky multiplier theorem*, Canad. J. Math., vol. XXVII (1975), 827–836.
15. Evans R., *Resolving Banach spaces*, Studia Math., **69** (1980), 91–107.
16. Gamelin T. W., *Uniform algebras*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1969.
17. Hirsberg B., *M-ideals in complex functions spaces and algebras*, Israel J. Math., **12** (1972), 133–146.
18. Niculescu C. P., *Cunningham projections associated to an Alfsen-Effros type order relation*. In *Proc. of The Tenth Conference on Operator Theory*, Bucharest, August 26 – September 5, 1985. Operator Theory, Advances and Applications vol. 24 (1987), 251–263, Birkhäuser-Verlag, Basel.
19. Niculescu C. P., *The centralizer associated to an Alfsen-Effros type order relation*, Analele Univ. Craiova; part I, vol. XIV (1986), 1–9; part II, vol. XV (1987), 5–13.
20. Niculescu C. P., *A commutative-extremal extension of Hilbert-Schmidt theorem*, Revue Roum. Math. Pures et Appl., **34**(1989), 247–261.
21. Niculescu C. P., *An introduction to commutative convexity*. In vol. *Order structures in Functional Analysis*, p. 83–115, Ed. Academiei, Bucharest, 1989 (Romanian).
22. Niculescu C. P., *Lectures on Alfsen-Effros Theory*. Part I: Facial structures. Preprint, 1990.
23. Niculescu C. P., Păltineanu G. and Vuza D. T., *Interpolation and Approximation from the M-Theory Point of View*. Preprint.
24. Orhon M., *Boolean algebras of commuting projections*, Math. Z., **183** (1983), 531–537.
25. Perdrizet F., *Espaces de Banach ordonnés et idéaux*, J. Math. Pures et Appl., **49** (1970), 61–98.
26. Schaefer H. H., *Banach lattices and positive operators*, Springer Verlag, Berlin, 1973.
27. Uttersrud U., *On M-ideals and the Alfsen-Effros structure topology*, Math. Scand., **43** (1978), 369–381.
28. Wickstead A. W., *The structure space of a Banach lattice*, J. Math. Pures et Appl., **56** (1977), 39–54.
29. Wils W., *The ideal center of partially ordered vector spaces*, Acta Math., **127** (1971), 41–77.